Journal of Applied Mechanics and Technical Physics, Vol. 37, No. 5, 1996

DIAGNOSIS OF THE PROPERTIES OF A STRUCTURIZED MEDIUM BY LONG NONLINEAR WAVES

V. A. Vakhnenko

UDC 532.59:517.19

It was traditionally believed that long-wave processes in inhomogeneous media can be simulated within the framework of a homogeneous medium. It is known [1–3] that, at the acoustic level, the structure of a medium for long waves can be taken into account by means of disperse-dissipative properties of a homogeneous medium. At the same time, the evolution of finite-amplitude long waves is directly affected by the medium structure, as is shown by a rigorous mathematical analysis using asymptotic averaging [4–6].

This paper proves that the effect of the structure on nonlinear long-wave perturbations is so significant that one can predict the properties of the medium from the features of evolution of the wave field.

1. Asymptotic Averaged Model. Media with regular structures are elementary inhomogeneous media for which the effect of the structure can be analyzed. The mechanism of propagation of long-wave perturbations is studied for a periodic medium with equalization of stresses and mass velocities on the boundaries of adjacent components. It is assumed that the microstructural element of the medium is sufficiently large that the laws of classical continuum mechanics can be applied to it. The medium is barotropic. We consider media in a hydrodynamic approximation ignoring shear stresses. The specific volume $V = \rho^{-1}$ and the sound velocity c are considered periodically varying properties of undisturbed media.

One method of the averaged description of media of regular structure is that of asymptotic averaging [7, 8]. It is used to simulate long waves in compressible media [4]. Taking into account the well-known results for the case of plane symmetry [4–6], we derive an averaged system of equations for one-dimensional motion of arbitrary symmetry.

The assumed equations of one-dimensional unsteady motion are the equations of motion for each individual component in Lagrangian variables:

$$\frac{\partial r^\nu}{\partial l^\nu} = \frac{V}{V_0}, \qquad u = \frac{\partial r}{\partial t}, \qquad \frac{\partial u}{\partial t} + V_0 \left(\frac{r}{l}\right)^{\nu-1} \frac{\partial p}{\partial l} = 0. \tag{1.1}$$

One can use the continuity equation in alternative form:

$$\frac{\partial V}{\partial t} - \nu V_0 \frac{\partial r^{\nu-1} u}{\partial l^\nu} = 0. \tag{1.2}$$

We use standard notation. The matching conditions are the equality of mass velocities and the equality of pressures at the boundaries of the components. For each component the equations of state are known:

$$dp = c^2 d\rho. \tag{1.3}$$

It is convenient to use dimensionless variables [9], in which case the resulting dimensionless equations do not differ in form from the assumed equations. Therefore, we shall assume that Eqs. (1.1)–(1.3) are written in dimensionless variables.

An averaged system of equations for media with a periodic structure can be derived analytically. In the case of cylindrical ($\nu = 2$) and spherical ($\nu = 3$) symmetries the layers have the corresponding symmetry.

Subbotin Institute of Geophysics, Academy of Sciences of Ukraine, Kiev 252054. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 37, No. 5, pp. 35–42, September–October, 1996. Original article submitted July 21, 1995.

It turns out (as will be seen below) that the restriction caused by the periodical properties of the medium can be eliminated.

Following the asymptotic averaging technique [7, 8], the spatial coordinate $m = l^\nu/V_0$ is split into slow s and fast ξ independent variables:

$$m = s + \varepsilon\xi, \qquad \frac{\partial}{\partial m} = \frac{\partial}{\partial s} + \varepsilon^{-1}\frac{\partial}{\partial \xi}.$$

The dimensionless structural period $\varepsilon = \varepsilon'/\lambda$ (the ratio of the structural period ε' to the wavelength λ) is a small parameter. The solutions for r, p, u, and V are sought as series in powers of ε:

$$r^\nu(m,t) = (r^\nu)^{(0)}(s,t,\xi) + \varepsilon(r^\nu)^{(1)}(s,t,\xi) + \varepsilon^2(r^\nu)^{(2)}(s,t,\xi) + \dots\ ,$$
$$V(m,t) = V^{(0)}(s,t,\xi) + \varepsilon V^{(1)}(s,t,\xi) + \varepsilon^2 V^{(2)}(s,t,\xi) + \dots$$

The functions $f^{(i)}(\xi)$ are considered singly-periodic with respect to ξ.

Following the procedure described thoroughly for the case of plane symmetry in [5] in an approximation of the order of $O(\varepsilon^{-1})$, we obtain

$$\frac{\partial (r^\nu)^{(0)}}{\partial \xi} = 0, \qquad \frac{\partial p^{(0)}}{\partial \xi} = 0, \qquad \frac{\partial\big((r^{\nu-1})^{(0)}u^{(0)}\big)}{\partial \xi} = 0. \tag{1.4}$$

Therefore, the mass velocity $u^{(0)}$, the pressure $p^{(0)}$, and the Eulerian coordinate $(r^\nu)^{(0)}$ are independent of the fast variable ξ.

By way of example, we write the equation of momentum in an approximation of the order of $O(\varepsilon^0)$:

$$\frac{\partial u^{(0)}}{\partial t} + \nu(r^\nu)^{(0)}\frac{\partial p^{(0)}}{\partial s} + \nu(r^\nu)^{(1)}\frac{\partial p^{(0)}}{\partial \xi} + \nu(r^\nu)^{(0)}\frac{\partial p^{(1)}}{\partial \xi} = 0. \tag{1.5}$$

We now apply the averaging procedure, which is only possible in Lagrangian mass coordinates, since in this case the period is independent of wave motion. By definition, $\langle\cdot\rangle = \int\limits_0^1 (\cdot)\,d\xi$. Here we use the normalization condition $\int\limits_0^1 d\xi = 1$. On the one hand, we have one of the desired equations

$$\frac{\partial u^{(0)}}{\partial t} + \nu(r^\nu)^{(0)}\frac{\partial p^{(0)}}{\partial s} = 0, \tag{1.6}$$

since $\langle \partial p^{(1)}/\partial\xi\rangle = 0$ in view of the periodicity of $p^{(1)}$ with respect to ξ. On the other hand, we obtain $\partial p^{(1)}/\partial \xi = 0$ by subtracting Eq. (1.6) from Eq. (1.5). Therefore, $p^{(1)}$ is also independent of ξ.

The remaining averaged equations are written in a similar way:

$$\frac{\partial (r^\nu)^{(0)}}{\partial s} = \langle V^{(0)}\rangle, \qquad u^{(0)} = \frac{\partial r^{(0)}}{\partial t}; \tag{1.7}$$

$$d\langle V^{(0)}\rangle = -\Big\langle \frac{(V^{(0)})^2}{c^2}\Big\rangle dp. \tag{1.8}$$

Equation (1.2) has the form

$$\frac{\partial \langle V^{(0)}\rangle}{\partial t} - \nu\frac{\partial (r^{\nu-1})^{(0)}u^{(0)}}{\partial s} = 0. \tag{1.9}$$

The nondependence of the variables on the fast coordinate in (1.4) means that $u^{(0)} = \langle u^{(0)}\rangle$, $p^{(0)} = \langle p^{(0)}\rangle$, and $r^{(0)} = \langle r^{(0)}\rangle$. Unlike these values, the specific volume $V^{(0)}(\xi)$ is a function of ξ. System (1.6)–(1.9) is intergo-differential. Further we restrict ourselves to a zero approximation with respect to ε and omit the superscript 0.

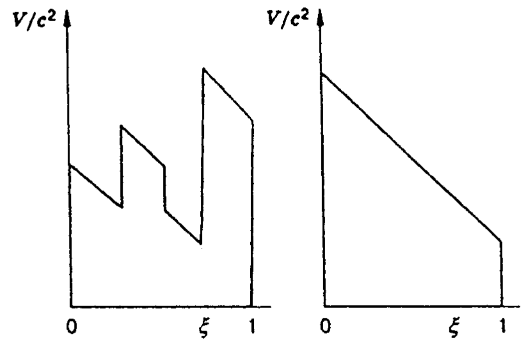

Fig. 1

Equations (1.6)–(1.9) are derived for a rigorously periodic medium. However, one can show that for media with a quasi-periodic structure these equations are also valid. Indeed, the pressure p and the mass velocity u are constant over the entire period of the structure. On the large scale of s, the effect of perturbations is manifested itself in wave motion of the medium, while on the microscale of ξ the effect is uniform (wave-free) throughout the structural period of the medium.

The behavior of the medium at the microlevel obeys thermodynamic laws only. At the macrolevel, the behavior is described by laws of wave dynamics (1.6) and (1.7) for the mean characteristics r, u, p, and $\langle V\rangle$. From a mathematical viewpoint in a zeroth approximation for ε the period size is infinitesimal ($\varepsilon \to 0$). This implies that the location of individual components in the period is of no significance. System (1.6)–(1.9) is invariant if the arrangement of the layers in the unit cell is changed or if the layers are split. Therefore, Eqs. (1.6)–(1.9) describe the behavior of any quasi-periodic (statistically inhomogeneous) medium that has the same mass content of components at the microstructure level irrespective of the location of the substance in the cell volume.

In nonlinear waves, the individual components are variously compressed. The structure of the medium changes, and this eventually results in a change in the averaged characteristics of the medium that differs from the change in the characteristics of a homogeneous medium under the same loading. This is the effect of the medium structure on the wave motion, although the equations of motion (1.6) and (1.7) are written for the averaged r, u, p, and $\langle V\rangle$.

As was noted in [5, 6], the structure affects the propagation of nonlinear long waves. The occurrence of nonlinear effects is due to the presence of the term $\langle V^2/c^2\rangle$ in equation of state (1.8). Introduction of the effective mean speed of sound by the formula

$$\tilde{c} = \sqrt{\langle V\rangle^2 \left\langle \frac{V^2}{c^2} \right\rangle^{-1}} \tag{1.10}$$

reduces the system to the traditional form. It is obvious that $\tilde{c}$ is not a mean characteristic, i.e., $\tilde{c}^2 \neq \langle c^2\rangle$. The structure of the medium makes a certain contribution to nonlinearity. Indeed, $\tilde{c}$ is a function of pressure in the general case, although for each component the speed of sound can be independent of pressure [$c \neq f(p)$]. At the same time in the acoustic approximation, as was mentioned in [5, 6], the pressure- and mass-velocity fields in periodic and homogeneous media coincide under certain conditions of matching of the medium's properties.

There are media in which V/c^2 does not change throughout the period. Under the action of wave perturbations, they behave as homogeneous media. The individual elements of the structure respond to the pressure change so that the relative structure remains constant, i.e., the ratio $V(\xi,p)/V(\xi,p_0)$ is independent of ξ. The averaged values $\langle V\rangle$ and $\langle V^2/c^2\rangle$ can be written as $\langle V\rangle = V/c^2\langle c^2\rangle$ and $\langle V^2/c^2\rangle = (V/c^2)^2\langle c^2\rangle$. The effective speed of sound (1.10) in this case is a mean characteristic: $\tilde{c}^2 = \langle c^2\rangle$, and, hence, the entire system of equations is representable in the mean variables u, p, $\langle V\rangle$, and $\tilde{c} = \sqrt{\langle c^2\rangle}$. For such media, their structure does not manifest itself in nonlinear wave motion.

2. Diagnosis of a Medium by Long Nonlinear Waves. The influence of the medium's structure on the wave field of long-wave nonlinear perturbations is proved in [5, 6]. However, the question arises of whether the information contained in the wave field is sufficient to restore the structure of the medium. The answer to this question turns out to be positive. Let us describe the method of determining the structure of the medium from the laws of motion of finite-amplitude pressure waves.

Let us prove that the effect of the structure on nonlinear long-wave perturbations is so great that the evolution of the field enables one to determine the properties of the medium, namely, the dependence of V/c^2 on the fast variable, i.e., the distribution of the value V/c^2 on the period of the structured medium. We consider plane waves. Unlike in [5, 6], the requirement of agreement of speeds of sound in all components is not imposed.

An important circumstance should be noted. Since in the asymptotic averaged model the structural period ε' is infinitesimal with respect to the wavelength λ, the location of the structural elements in the period cannot be determined exactly in the proposed diagnosis method. Thus, two structures that differ in the functional dependence of V/c^2 on ξ (for example, as in Fig. 1) affect wave motion in the same way. Consequently, these two media cannot be distinguished by long waves. Bearing in mind this constraint, we shall further assume for definiteness that the dependence of V/c^2 on the fast Eulerian coordinate ζ is a decreasing integrable one-to-one function on the segment $\zeta \in [0, 1]$, and beyond the segment it is zero. The variable ζ is determined by the relation $(\partial\xi/\partial\zeta)_t = \rho(\xi)$. The relation is analogous to the ratio between the commonly used Eulerian coordinate x and the Lagrangian mass coordinate m $(\partial m/\partial x)_t = \rho$.

Let us use the known fact from the probability theory [10]. The distribution function $f(x)$ (single-valued integrable positive function) is expressed in terms of its central moments $\alpha_n = \int\limits_{-\infty}^{\infty} x^n f(x)dx$ by means of inverse Fourier transform:

$$f(x) = F^{-1}\Big[\sum_{n=0}^{\infty} \alpha_n i^n \frac{q^n}{n!}\Big](x). \tag{2.1}$$

Let us consider the chain of transformations

$$\langle V(V/c^2)^n\rangle = \int\limits_0^1 V(\xi)\Big(\frac{V}{c^2}\Big)^n d\xi = \int\limits_0^1 V\Big(\frac{V}{c^2}\Big)^n \rho d\zeta$$

$$= \langle V\rangle \int\limits_{-\infty}^{\infty} \Big(\frac{V}{c^2}\Big)^n \frac{d\zeta}{d(V/c^2)}\, d(V/c^2) = n\langle V\rangle \int\limits_{-\infty}^{\infty} \Big(\frac{V}{c^2}\Big)^{n-1} \zeta d(V/c^2) = n\langle V\rangle \alpha_{n-1},$$

i.e., the central moment of the dependence of ζ on V/c^2 is expressed in terms of $\langle V(V/c^2)^n\rangle$. Finally, we find a function that is inverse to the desired one:

$$\zeta = F^{-1}\Big[\sum_{n=0}^{\infty} \frac{\langle V(Vc^{-2})^n\rangle}{(n+1)!\langle V\rangle} i^n q^n\Big]. \tag{2.2}$$

The coefficients $\langle V(Vc^{-2})^n\rangle$ $(n = 2, 3, \ldots)$ for this formula are easily calculated if the functional dependence of $\langle V\rangle$ on p or $\langle V^2/c^2\rangle$ of p is known. They are sequentially determined from the recursive relation

$$\frac{d\langle V(Vc^{-2})^n\rangle}{dp} = -(n+1)\langle V(Vc^{-2})^{n+1}\rangle, \tag{2.3}$$

which follows directly from the equation of state; (2.3) is the basic relation used in the new diagnosis method in which the properties of individual elements of a structurized medium are determined by means of nonlinear long waves.

The proposed method involves finding the coefficients $\langle V(Vc^{-2})^n\rangle$ of the power series, which can be determined from the laws of evolution of wave fields. The advantages of the diagnosis using wave action are particularly evident for media of complicated structure, including the environment.

The functional dependence of $\langle V\rangle$ on p can be determined in experiments on shock-wave propagation. As a result, the shock-wave velocity in the Lagrangian mass coordinates $D = ds/dt$ (D, kg/sec) and(or) the mass velocity u_1, and also the pressure p_1 behind the shock wave are found. The quantity $\langle V_1\rangle$ is calculated from the relations at shock discontinuity

$$D = \sqrt{(p_1 - p_0)/(\langle V_0\rangle - \langle V_1\rangle)}, \quad u_1 - u_0 = \sqrt{(p_1 - p_0)(\langle V_0\rangle - \langle V_1\rangle)},$$

which follow from the averaged equations. Measuring shock-wave parameters for different pressures p_1 we obtain the dependence $\langle V\rangle = \langle V\rangle(p)$. Then the recursive relation (2.3) is used to find $\langle V(Vc^{-2})^n\rangle$ for $n \geqslant 1$.

A self-similar rarefaction wave can be considered as a unique tool for determining the coefficients $\langle V(Vc^{-2})^n\rangle$. The self-similarity of motion of a rarefaction wave, as follows from (2.5) in [6], gives the propagation velocities ds/dt of individual parts of the wave profile at different pressures. This determines uniquely the dependence $\langle V^2/c^2\rangle = \langle V^2/c^2\rangle(p)$, and the dependences of $\langle V(Vc^{-2})^n\rangle$ for $n \geqslant 2$ are found from (2.3). We have the most accurate values of the coefficients at pressures at which the rarefaction-wave profile has the greatest inflection (see Fig. 1 in [6]), i.e., the perturbation is considerably affected by the structure. Therefore, in this pressure range the structure of the medium can be determined most exactly.

3. Approximation of the Medium to be Diagnosed by a Layered Medium. Diagnosis of the properties of a structurized medium by long nonlinear waves is concerned with determining the values of $\langle V(Vc^{-2})^n\rangle$ from experimental results. Naturally, the question arises of the accuracy of description of the structure of the medium by restricted series (2.2). Let us show that the partial sum of series (2.2) approximates the desired function $\zeta = \zeta(V/c^2)$ by a step function, i.e., the initial medium will be approximated by a layered one.

We write a sequence of identities for an arbitrary integrable function:

$$2\pi f(-x) = F\big[F[f(x)](q)\big](x) = F\Big[\sum_{n=0}^{\infty} \frac{i^n q^n}{n!}\alpha_n\Big] = \sum_{n=0}^{\infty} \frac{i^n \alpha_n}{n!} 2\pi(-i)^n \delta^{(n)}(x).$$

Here we used the well-known relations for the Fourier transform [11] and (2.1). Therefore, the arbitrary integrable function is representable as a series:

$$f(-x) = \sum_{n=0}^{\infty} \frac{\alpha_n}{n!}\delta^{(n)}(x). \tag{3.1}$$

Let us consider the step function $f_1(x)$ comprising N steps

$$f_1(x) = \begin{cases} \varphi_1, & \dots \quad 0 < x \leqslant b_1, \\ \varphi_2, & \dots \quad b_1 < x \leqslant b_2, \\ \vdots & \ddots \qquad \vdots \\ \varphi_N, & \dots \quad b_{N-1} < x \leqslant b_N. \end{cases}$$

Using this function, we approximate the unknown function $f(x)$. It is obvious that, increasing the number of steps N and selecting the values of φ_i and b_i, one can approximate any integrable function $f(x)$ by the step function $f_1(x)$. It is convenient to use the form

$$f_1(-x) = \varphi_1[\Theta(x+b_1) - \Theta(x)] + \varphi_2[\Theta(x+b_2) - \Theta(x+b_1)] + \ldots + \varphi_N[\Theta(x+b_N) - \Theta(x+b_{N-1})]. \tag{3.2}$$

We expand the Heaviside formula $\Theta(x+b)$ in its Taylor series expansion in the vicinity of point x:

$$\Theta(x+b) = \Theta(x) + \sum_{n=1}^{\infty} \frac{b^n}{n!}\Theta^{(n)}(x).$$

Equating functions (3.1) and (3.2) and assuming that the number of steps in the function $f_1(x)$ is infinitely great, we obtain

$$\varphi_1 \sum_{n=0}^{\infty} \frac{b_1^{n+1}}{(n+1)!}\delta^{(n)}(x) + \varphi_2 \sum_{n=0}^{\infty} \frac{b_2^{n+1} - b_1^{n+1}}{(n+1)!}\delta^{(n)}(x) + \ldots$$

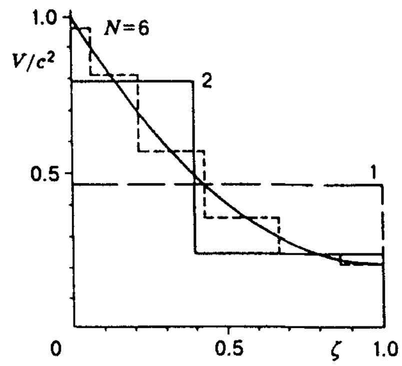

Fig. 2

$$+\varphi_N \sum_{n=0}^{\infty} \frac{b_N^{n+1} - b_{N-1}^{n+1}}{(n+1)!}\delta^{(n)}(x) + \ldots = \sum_{n=0}^{\infty} \frac{\alpha_n}{n!}\delta^{(n)}(x). \tag{3.3}$$

This relation shows that if one uses the partial sum $\sum_{n=0}^{2N-1}(\alpha_n/n!)\delta^{(n)}(x)$ on the right-hand side of series (3.3) and N first terms from the left-hand side, the desired function $f(x)$ is approximated by the step function $f_1(x)$, the number of steps being equal to N. In other words, if it is required to simulate the structure of the medium by means of N periodically repeated layers, one should know $2N-1$ moments of α_n, i.e., the values of $\langle V(Vc^{-2})^n\rangle$.

For convenience we expand (3.3). For this, we multiply it by x^n and integrate over the entire x axis. We obtain a nonlinear system of equations with respect to unknowns $b_1, b_2, \ldots, b_N$, and $\varphi_2, \varphi_3, \ldots, \varphi_N$ ($\varphi_1 = 1$ by virtue of normalization)

$$\begin{aligned}
&\varphi_1 b_1 + \varphi_2(b_2 - b_1) + \varphi_3(b_3 - b_2) + \ldots + \varphi_N(b_N - b_{N-1}) = \alpha_1,\\
&\varphi_1 b_1^2 + \varphi_2(b_2^2 - b_1^2) + \varphi_3(b_3^2 - b_2^2) + \ldots + \varphi_N(b_N^2 - b_{N-1}^2) = 2\alpha_2,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\varphi_1 b_1^{2N-1} + \varphi_2(b_2^{2N-1} - b_1^{2N-1}) + \varphi_3(b_3^{2N-1} - b_2^{2N-1}) + \ldots + \varphi_N(b_N^{2N-1} - b_{N-1}^{2N-1}) = (2N-1)\alpha_{2N-1}.
\end{aligned} \tag{3.4}$$

If now b_i is the partition of $(V/c^2)_i$, and φ_i the partition of ζ_i, we obtain a system of equations for determining the structure of the medium. Solution of this system of equations provides information on the properties of the medium, i.e., V/c^2 in the period of the structure is found as a step function.

Note a particular case of a periodic medium for which the value of V/c^2 is constant at the period. The propagation of long nonlinear waves in this medium, as was mentioned above, does not differ from that in a homogeneous medium. The same result follows from system (3.4). Indeed, for a homogeneous medium, the moments $\alpha_n = \int_{-\infty}^{\infty} x^{n-1}dx = 1/n$. Therefore, only unities are on the right-hand side of the system. Evidently, the solution of the system is of the form $b_1 = b_2 = \ldots = b_N = 1$ and $\varphi_1 = 1$ (any φ_i for $i \geqslant 2$), and it corresponds to a layered medium for which $V/c^2 \neq f(\zeta)$; in particular, this medium can be homogeneous.

By way of example, Fig. 2 presents the calculation results for a layered medium that describes the preassigned medium $V/c^2 = 0.2 + 0.8(1-\zeta)^2$ most adequately. This implies that $2N-1$ averaged characteristics $\langle V(Vc^{-2})^n\rangle$ correspond to N layers in the medium diagnosed and in the layered medium.

Thus, based on an asymptotic averaged model for a structurized medium, a new method for diagnosing the properties of the medium by long nonlinear waves is proposed.

The authors are thankful to Corresponding Member of the Academy of Sciences of Ukraine V. A. Danilenko for helpful and fruitful discussion of the results.

The work was partially supported by the International Science Foundation (Grant UAE200).

REFERENCES

1. L. M. Brekhovskikh and O. A. Godin, *Acoustics of Layered Media* [in Russian], Nauka, Moscow (1989).
2. V. A. Krasil'nikov and V. V. Krylov, *Introduction to Physical Acoustics* [in Russian], Nauka, Moscow (1984).
3. K. Aki and P. Richards, *Qualitative Seismology: Theory and Methods* [Russian translation], Nauka, Moscow (1983), Vols. 1 and 2.
4. N. S. Bakhvalov and M. É. Églit, "Processes not described in terms of average characteristics in periodic media," *Dokl. Akad. Nauk SSSR*, **268**, No. 4, 836–840 (1983).
5. V. A. Vakhnenko and V. V. Kulich, "Long-wave processes in a periodic medium," *Prikl. Mekh. Tekh. Fiz.*, No. 6, 49–56 (1992).
6. V. A. Vakhnenko, V. A. Danilenko, and V. V. Kulich, "Averaged description of shock-wave processes in periodic media," *Khim. Fiz.*, **12**, No. 3, 383–389 (1993).
7. N. S. Bakhvalov and G. P. Panasenko, *Process Averaging in Periodic Media* [in Russian], Nauka, Moscow (1984).
8. E. Sanchez-Palencia, *Inhomogeneous Media and Vibration Theory*, Springer Verlag, New York (1980).
9. N. S. Bakhvalov and M. É. Églit, "On the propagation velocity of perturbations in microinhomogeneous elastic media with low shear elasticity," *Dokl. Ross. Akad. Nauk*, **323**, No. 1, 13–18 (1992).
10. G. Korn and T. Korn, Mathematical Handbook for Scientists and Engineers, McGraw-Hill, New York (1968).
11. Yu. A. Brychkov and A. P. Prudnikov, *Integral Transformations of Generalized Functions* [in Russian], Nauka, Moscow (1977).